2. I. I. Trunin, "One variant of the equation of state in creep," in: Deformation and Fracture of Solids [in Russian], Izd. Mosk. Gos. Univ., Moscow (1977).
3. G. F. Lepin, Creep of Metals and Creep Resistance Criteria [in Russian], Metallurgiya, Moscow (1976).
4. O. V. Sosnin, "Evaluation of rupture strength from dissipative characteristics," in: Laws of Creep and Rupture Strength [in Russian], Mashinostroenie, Moscow (1983).
5. B. V. Gorev, A. F. Nikitenko, and O. V. Sosnin, "Substantiation of an energy variant of creep theory," Probl. Prochn., No. 11 (1976).
6. O. V. Sosnin and I. K. Shokalo, "Energy variant of the theory of creep and rupture strength," Probl. Prochn., No. 1 (1974).
7. Yu. N. Rabotnov, Creep of Structural Elements [in Russian], Nauka, Moscow (1966).
8. A. L. Arshakuni, G. P. Mel'nikov, et al., "Experimental verification of models of transient creep," Probl. Prochn., No. 4 (1981).
9. V. N. Kiselevskii and B. D. Kosov, "Experimental verification of the equation of state for the creep of a strain-hardening material," Probl. Prochn., No. 9 (1976).

DESIGN AND LAYOUT OF LAYERED PLATES

A. G. Kolpakov

UDC 639.3

Calculation of the stiffness characteristics of layered plates is carried out on the basis of asymptotic studies for the problem of elasticity theory in thin (thickness $h \to 0$) regions [1]. Application to the equations obtained of methods in [2, 3] made it possible to solve the problem of designing layered plates with a prescribed set of stiffness characteristics.

Characteristic Equations for Layered Plates. An asymptotic analysis was provided in [1] for the problem of elasticity theory in a thin region whose thickness h tends toward zero, and two methods were proved which may be used in designing layered plates: a limiting transition with $h \to 0$ and the same limiting transition invoking a cellular problem (problem L in the terms of [1]). In the first case we obtain explicit equations for calculating stiffness and, in the second, the same equations but with prior solution of the cellular problem. In this work we follow the second path in studying the mechanics of layered plate bending.

Let the plate in question be formed of layers of uniform isotropic materials (parallel planes Ox_1x_2). Plate thickness $h \ll 1$. We cover the plate with a rectangular network with a side ~h long. An element of this network P_h separates a cell $Y_h = P_h \times [-h/2, h/2]$, called the cellular periodicity. In variables $y = 2x/h$ a cell of periodicity Y_h is converted into

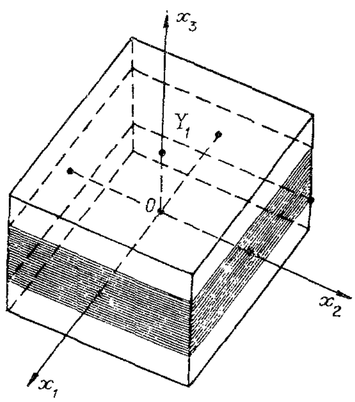

Fig. 1

Novosibirsk. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 4, pp. 152-161, July-August, 1989. Original article submitted February 9, 1988.

a cell $Y_1 = P_1 \times [-1, 1]$ (Fig. 1). We shall assume that $P_1 = [-1, 1] \times [-1, 1]$. The following characteristic equations were obtained in [1] for thin plates:

$$N_{ij} = S^{11}_{ij\alpha\beta}\gamma_{\alpha\beta} - S^{12}_{ij\alpha\beta}\frac{\partial^2 w}{\partial x_\alpha \partial x_\beta}; \tag{1}$$

$$M_{ij} = S^{21}_{ij\alpha\beta}\gamma_{\alpha\beta} - S^{22}_{ij\alpha\beta}\frac{\partial^2 w}{\partial x_\alpha \partial x_\beta}, \quad \alpha = 1, 2, \tag{2}$$

where

$$S^{\mu\nu}_{ij\alpha\beta} = (-1)^{\mu+\nu}\int_{P_1} (-y_3)^{\mu-1}\left[(-y_3)^{\nu-1}a_{ij\alpha\beta} + a_{ijkl}\frac{\partial\xi^{\alpha\beta\nu}_k}{\partial y_l}\right]d\mathbf{y}, \tag{3}$$

$\mu, \nu = 1, 2$, and function $\xi^{\alpha\beta\nu}$ is determined from the cellular problem (problem L in [1]) in cell Y_1

$$\frac{\partial}{\partial y_j} a_{ijkl}(\mathbf{y}) \frac{\partial}{\partial y_l}(\xi^{\alpha\beta\nu} + \mathbf{e}_\alpha y_\beta y_3^{\nu-1})_k = 0 \tag{4}$$

with zero normal stresses at the plate surfaces with $y_3 = \pm 1/2$, a condition of periodicity at P_1 (with respect to variables y_1 and y_2), and normalizing $\langle\xi^{\alpha\beta\nu}\rangle = 0$ ($\langle\ \rangle$ signifies average for cell Y_1). In (1) and (2), $\{N_{ij}\}$ and $\{\gamma_{ij}\}$ are forces and strains in the plane of the plate (in their classical sense); $\{M_{ij}\}$ are moments; $\mathbf{v}$ are displacements in the plane of the plate; w is normal deflection.

We consider problem (4) for a particular case of $\alpha = 1$, $k = 1$. As is easy to see, it is a problem of extension ($\nu = 1$) or bending ($\nu = 2$) for a layered body Y_1. We shall find its solution in the form

$$\sigma_{11} = \sigma_{11}(y_3),\ \sigma_{22} = \sigma_{22}(y_3),\ \sigma_{ij} = 0 \text{ with } ij \neq 11, 22,\ e_{22} = 0 \tag{5}$$

[$\sigma_{ij} = a_{ijk\ell}(\mathbf{y})e_{k\ell}$ are stresses, $e_{ij} = (1/2)(u_{i,j} + u_{j,i})$ are strains corresponding to displacements $\mathbf{u}^{k\alpha\nu} = \xi^{k\alpha\nu} + \mathbf{e}_k y_\alpha y_3{}^{\nu-1}$ from problem (4)]. As can be seen, in seeking a solution in the form of (5) conditions for normal stresses with $y_3 = \pm 1/2$ are fulfilled.

With $\nu = 1$, we find the solution of problem (4) in the form

$$\sigma_{11} = \frac{E(y_3)}{1-\nu^2(y_3)}, \quad \sigma_{22} = \frac{E(y_3)\nu(y_3)}{1-\nu^2(y_3)} \tag{6}$$

or

$$u_1^{111} = y_1,\ u_2^{111} = 0,\ u_3^{111} = -\int\frac{\nu(y_3)}{1-\nu(y_3)}dy_3 + \text{const.}$$

Conditions of periodicity for functions $\xi^{111} = \mathbf{u}^{111} - \mathbf{e}_1 y_1$ are fulfilled.

With $\nu = 2$ again we find a solution in the form of (5). From Hooke's law we have

$$e_{11} = \frac{\sigma_{11}}{E(y_3)} - \frac{\nu(y_3)\sigma_{22}}{E(y_3)},\ e_{22} = \frac{\sigma_{22}}{E(y_3)} - \frac{\nu(y_3)\sigma_{11}}{E(y_3)},\ e_{33} = -\frac{\nu(y_3)}{E(y_3)}(\sigma_{11} + \sigma_{22}), \tag{7}$$

in view of which $\sigma_{22} = \nu(y_3)\sigma_{11}$ and

$$e_{11} = \frac{1-\nu^2(y_3)}{E(y_3)}\sigma_{11}, \quad e_{33} = -\frac{\nu(y_3)(1+\nu(y_3))}{E(y_3)}\sigma_{11}, \tag{8}$$

whence (taking account of symmetry of the problem with respect to plane $y_1 = 0$) by integrating (8) we obtain

$$\begin{gathered} u_1^{112} = \frac{1-\nu^2(y_3)}{E(y_3)}\sigma_{11}(y_3)y_1, \quad u_2^{112} = 0, \\ u_3^{112} = -\int\frac{\nu(y_3)(1+\nu(y_3))}{E(y_3)}\sigma_{11}(y_3)dy_3 + U(y_1). \end{gathered} \tag{9}$$

In view of $e_{13} = 0$ from (9), we find that

$$\frac{\partial u_1}{\partial y_3} + \frac{\partial u_3}{\partial y_1} = \frac{\partial}{\partial y_3}\left(\frac{1-\nu^2}{E}\sigma_{11}\right)(y_3)\, y_1 + \frac{\partial U(y_1)}{\partial y_1} = 0. \tag{10}$$

Equality (10) is only possible in the case

$$\frac{\partial}{\partial y_3}\left(\frac{1-\nu^2}{E}\sigma_{11}\right)(y_3) = C = \text{const}, \quad \frac{\partial U(y_1)}{\partial y_1} = -Cy_1, \tag{11}$$

whence

$$\frac{1-\nu^2(y_3)}{E(y_3)}\sigma_{11}(y_3) = Cy_3 + A, \quad U(y_1) = -\frac{C}{2}y_1^2 + B. \tag{12}$$

After which, from (9) we have

$$\begin{gathered} u_1^{112} = Cy_1y_3 + Ay_1, \quad u_2^{112} = 0, \\ u_3^{112} = -\int \frac{\nu(y_3)}{1-\nu(y_3)}(Cy_3 + A)\, dy_3 - \frac{C}{2}y_1^2 + B. \end{gathered} \tag{13}$$

So that function $\xi^{112} = u^{112} - e_1y_1y_3$ [u^{112} gives (13)] is periodic with respect to y_1 and y_2, $C = 1$ and $A = 0$ should be placed in (13). Constant B, which may be determined from the condition $\langle\xi^{112}\rangle = 0$, does not affect the value of $S_{ijk\ell}^{\mu\nu}$, in view of which it is not determined. As a result of this we obtain

$$\sigma_{11} = \frac{E(y_3)\, y_3}{1-\nu^2(y_3)}, \quad \sigma_{22} = \frac{E(y_3)\,\nu(y_3)\, y_3}{1-\nu^2(y_3)}. \tag{14}$$

Substitution of expressions (6) and (14) in (3) leads to equations for calculating stiffness $S_{iiii}^{\mu\nu}$ ($i = 1, 2$):

$$\begin{gathered} S_{iiii}^{11} = \frac{h}{2}\int_{-1}^{1} \frac{E(y_3)}{1-\nu^2(y_3)}\, dy_3, \\ S_{iiii}^{12} = S_{iiii}^{21} = \left(\frac{h}{2}\right)^2 \int_{-1}^{1} \frac{E(y_3)\, y_3}{1-\nu^2(y_3)}\, dy_3, \quad S_{iiii}^{22} = \left(\frac{h}{2}\right)^3 \int_{-1}^{1} \frac{E(y_3)\, y_3^2}{1-\nu^2(y_3)}\, dy_3. \end{gathered} \tag{15}$$

We calculate $S_{1122}^{\mu\nu} = S_{2211}^{\mu\nu}$. For this it is necessary to know, for example, stresses σ_{22} corresponding to functions $\xi^{11\nu} + e_1y_1y_2^{\nu-1}$ ($\alpha = 1$, $k = 1$). These stresses are known [see (6) and (14)]. As a result of this, from (6), (14), and (3) we have

$$\begin{gathered} S_{1122}^{11} = S_{2211}^{11} = \frac{h}{2}\int_{-1}^{1} \frac{E(y_3)\,\nu(y_3)}{1-\nu^2(y_3)}\, dy_3, \\ S_{1122}^{12} = S_{1122}^{21} = S_{2211}^{12} = S_{2211}^{21} = \left(\frac{h}{2}\right)^2 \int_{-1}^{1} \frac{E(y_3)\,\nu(y_3)\, y_3}{1-\nu^2(y_3)}\, dy_3, \\ S_{1122}^{22} = S_{2211}^{22} = \left(\frac{h}{2}\right)^3 \int_{-1}^{1} \frac{E(y_3)\,\nu(y_3)\, y_3^2}{1-\nu^2(y_3)}\, dy_3. \end{gathered} \tag{16}$$

In order to calculate stiffness $S_{1212}^{\mu\nu}$, a solution of problem (4) should be obtained for a function of the form $u^{12\nu} = \xi^{12\nu} + e_1y_2y_3^{\nu-1}$ ($k = 2$, $\alpha = 1$). With this aim it is possible to use the solution found above (with $k = 1$, $\alpha = 1$) in a coordinate system turned through 45°. Thus, if function $v = u^{22\nu} - u^{11\nu}$ is considered in a coordinate system turned through 45°, then, after returning to the original coordinate system, function v will be the sum of the solutions of problem (4): $\xi^{12\nu} + e_1y_2y_3 + \xi^{21\nu} + e_2y_1y_3$ (the sum of the solutions relating to $k = 2$, $\alpha = 1$ and $k = 1$, $\alpha = 2$). After which it is easy to find stresses σ_{ij} corresponding to this function v:

$$\begin{gathered} \sigma_{12} = \sigma_{21} = E(y_3)/(1 + \nu(y_3)),\ \sigma_{ij} = 0 \text{ with } ij \neq 12, 21\ (\nu = 1), \\ \sigma_{12} = \sigma_{21} = E(y_3)y_3/(1 + \nu(y_3)),\ \sigma_{ij} = 0 \text{ with } ij \neq 12, 21\ (\nu = 2) \end{gathered}$$

and values

$$S^{11}_{1212} = \frac{h}{2}\int\limits_{-1}^{1}\frac{E(y_3)}{1+\nu(y_3)}\,dy_3,$$

$$S^{12}_{1212} = S^{21}_{1212} = \left(\frac{h}{2}\right)^2\int\limits_{-1}^{1}\frac{E(y_3)\,y_3}{1+\nu(y_3)}\,dy_3, \tag{17}$$

$$S^{22}_{1212} = \left(\frac{h}{2}\right)^3\int\limits_{-1}^{1}\frac{E(y_3)\,y_3^2}{1+\nu(y_3)}\,dy_3.$$

Not shown explicitly $S^{\mu\nu}_{ij\alpha\beta}$ equal zero (with an accuracy to known symmetries).

Note 1. Equations (15)-(17) for calculating plate stiffness in tension $\{S^{11}_{ij\alpha\beta}\}$ and bending $\{S^{22}_{ij\alpha\beta}\}$ coincide with equations obtained in [4] for the same characteristics of a plate containing a large number of layers (averaged in a classical sense). A model for an undeformed normal with a condition of equality to zero of normal stresses in the plate should be used in [4].

Note 2. With $E(y_3)$ = const, $\nu(y_3)$ = const classical equations emerge from (15)-(17) for the stiffness of uniform plates in tension and bending.

Design of Layered Plates with a Prescribed Set of Stiffnesses. The following problem is posed (design problem): a) by arranging a prescribed set of uniform isotropic materials to solve the question of the possibility of creating on the basis of the materials used a layered plate with a prescribed set of stiffnesses $\{S^{\mu\nu}_{ij\alpha\beta}\}$, b) if the question is solved positively, to indicate a method for creating (designing) this plate.

By returning to Eqs. (15)-(17) it is noted that solution of the design problem formulated is reduced to solving the equation

$$\left(\frac{h}{2}\right)^{\mu+\nu-1}\int\limits_{-1}^{1}E(y_3)\,\Phi_{ij\alpha\beta}(\nu(y_3))\,y_3^{\mu+\nu-2}\,dy_3 = S^{\mu\nu}_{ij\alpha\beta} \tag{18}$$

with respect to functions $E(y_3)$ and $\nu(y_3)$ and number h [functions $\Phi_{ij\alpha\beta}(\nu)$ are determined in (15)-(17)]. Apparently the values listed clearly describe a plate (its thickness and structure through the thickness). Problem (18) is incorrect in [5].

Cases ν = const and $E_0 \le E(y) \le E^0$ are of theoretical interest since it is possible to obtain analytically the condition for solving problem (18) and based on it to describe (analytically) numerous possible stiffness values for layered plates.

It is easy to note that with $\nu(y_3)$ = const = ν all $\{S^{\mu\nu}_{ij\alpha\beta}\}$ are expressed in terms of three functionals:

$$I_i(E) = \int\limits_{-1}^{1}E(y)\,y^i\,dy, \quad i = 0, 1, 2. \tag{19}$$

Let $E(y) \in \{f \in L_\infty([-1, 1]) : E_0 \le E(y) \le E^0\} \equiv U_E$ is a class of materials used in plate design. It is evident that problem (18) is solvable when, and only when, $\{S^{\mu\nu}_{ij\alpha\beta}\}$ pertains to the image of set U_E with a representation, prescribed by the right-hand part of (18), for whose computation it is sufficient to know image U_E with representation (19).

We consider the extreme problem

$$I_j(E) = A_j,\ j = 0, \ldots, i,\ i = 1, 2; \tag{20}$$

$$I_{i+1}(E) \to \min = \min_{i+1}(\max = \max_{i+1}); \tag{21}$$

$$E(y) \in U_E. \tag{22}$$

LEMMA 1. Functional $I_{i+1}(E)$ $(i = 0, 1)$ in set U_E with fulfillment of condition (20) takes all of the values from the range $[\min_{i+1}, \max_{i+1}]$.

Proof. Let $E_1(y)$, $E_2(y) \in U_E$ be solutions of maximization and minimization problems (20)-(22) with given i. Since set U_E is convex, and functional $I_i(E)$ (19) are linear, then function $E(y) = \lambda E_1(y) + (1 - \lambda)E_2(y) \in U_E$ and it satisfies equality (20) with any $\lambda \in [0, 1]$. For it we have $I_{i+1}(E) = \lambda \max_{i+1} + (1 - \lambda)\min_{i+1}$, whence we obtain what is required.

LEMMA 2. Functional $I_0(E)$ in set U_E takes all the values of the range $[2E_0, 2E^0]$.

This is evident.

LEMMA 3. With i = 0 in problem (20)-(22),

$$\min_1 = \frac{(2E_0 - A_0)(2E^0 - A_0)}{2(E^0 - E_0)}, \quad \max_1 = -\frac{(2E_0 - A_0)(2E^0 - A_0)}{2(E^0 - E_0)}. \tag{23}$$

Proof. Expressions (20)-(22) are a Lyapunov problem [6] as a result of which its solution $\hat{E}(y) \in U_E$, if it exists, satisfies the maximum principle [6, p. 354] having, in the case in question, a form: for almost all $y \in [-1, 1]$

$$\min (\lambda_0 y E(y) + \lambda_1 E(y)) = \lambda_0 y \widehat{E}(y) + \lambda_1 E(y). \tag{24}$$

A minimum is taken for all $E(y) \in U_E$ satisfying condition (20) (with i = 0). In view of the linearity of (24) with respect to E(y) and $\hat{E}(y)$ we find that $\hat{E}(y)$ for almost all $y \in [-1, 1]$ takes two values: either E_0 or E^0. After which finding function $\hat{E}(y)$ and obtaining Eq. (23) is trivial.

LEMMA 4. With i = 1 in problem (20)-(22),

$$\min_2 = \frac{2E_0}{3} - \frac{(2E_0 - A_0)^3}{3(E^0 - E_0)^2}, \quad \max_2 = \frac{2E^0}{3} - \frac{(2E^0 - A_0)^3}{3(E^0 - E_0)^2}. \tag{25}$$

Proof. In order to solve $\hat{E}(y)$ of problem (20)-(22) with i = 1, the minimum principle [6] takes the form: for almost all $y \in [-1, 1]$,

$$\min (\lambda_0 y^2 E(y) + \lambda_1 y E(y) + \lambda_2 E(y)) = \lambda_0 y^2 \widehat{E}(y) + \lambda_1 y \widehat{E}(y) + \lambda_2 \widehat{E}(y). \tag{26}$$

A minimum is taken with respect to $E(y) \in U_E$ satisfying (20) (with i = 1). In view of linearity for expressions in (26) with respect to E(y) and $\hat{E}(y)$, we obtain

$$\widehat{E}(y) = \begin{cases} E_0 \text{ with } \lambda_0 y^2 + \lambda_1 y + \lambda_2 \leqslant 0, \\ E^0 \text{ with } \lambda_0 y^2 + \lambda_1 y + \lambda_2 > 0. \end{cases} \tag{27}$$

As a result of the fact that in the right-hand part of (27) there is a quadratic trinomial, function $\hat{E}(y) = E_0$ with $y \in [a, b] \subset [-1, 1]$ and $\hat{E}(y) = E^0$ with $y \notin [a, b]$ (in the case of $\lambda_0 > 0$). It is easy to prove that in selecting $\lambda_0 = 1$ and

$$2a = -\frac{2A_1}{2E_0 - A_0} - \frac{2E_0 - A_0}{E^0 - E_0}, \quad 2b = -\frac{2E_0 - A_0}{E^0 - E_0} - \frac{2A_1}{2E_0 - A_0} \tag{28}$$

conditions (20) with i = 1 are satisfied, and $-1 \leq a \leq b \leq 1$ (for A_0 and A_1 existing within the limits indicated in Lemmas 2 and 3), whence Eq. (25) emerges.

Lemmas 2-4 give a description of an image of set U_E with representation (19), after which it is possible to obtain an image of set U_E with a representation described by the right-hand part of (18). We give this description in mechanical terms.

Proposition. A. A layered structure plate of thickness h prepared from materials with Young's moduli E (ν = const) satisfying the condition $E_0 \leq E \leq E^0$, i.e., $E(y) \in U_E$, may exhibit a set of stiffnesses S^{11}_{1111}, S^{12}_{1111}, and S^{22}_{1111} (and it may not exhibit others):

$$\frac{hE_0}{1-\nu^2} \leqslant S^{11}_{1111} \leqslant \frac{hE^0}{1-\nu^2},$$

$$\frac{h^2(2E_0 - A_0)(2E^0 - A_0)}{8(E^0 - E_0)(1-\nu^2)} \leqslant S^{12}_{1111} \leqslant -\frac{h^2(2E_0 - A_0)(2E^0 - A_0)}{8(E^0 - E_0)(1-\nu^2)},$$

$$\frac{h^3}{24(1-\nu^2)}\left(2E_0-\frac{(2E_0-A_0)^3}{3(E^0-E_0)^2}\right)\leqslant S_{1111}^{22}\leqslant\frac{h^3}{24(1-\nu^2)}\left(2E^0-\frac{(2E^0-A_0)^3}{3(E^0-E_0)^2}\right), \tag{29}$$

where $A_0 = S_{1111}^{11}(1 - \nu^2)/h$. Nonlinear $S_{ij\alpha\beta}^{\mu\nu}$, not shown in (29), are expressed in terms of S_{1111}^{11}, S_{1111}^{12}, S_{1111}^{22} according to equations obtained from (15)-(17) with ν = const.

B. A plate with any permissible set of stiffnesses $S_{ij\alpha\beta}^{\mu\nu}$ (29) may be formed of not more than five layers using not more than three different materials.

Proof. Confirmation of A) follows from Lemmas 2-4. In order to prove the assertion in B), it is sufficient to use Lemma 1 in view of the fact that functions $E_1(y)$ and $E_2(y)$ in this lemma may be taken in the form of (27).

Discrete Design Problem. The limitations $E_0 \leq E \leq E^0$ used above are of interest from a theoretical viewpoint by supplying the possibility of evaluating a set of values of stiffness characteristics for layered plates which are "possible in principle." In practice limitations in the form $E_0 \leq E \leq E^0$, $\nu_0 \leq \nu \leq \nu^0$ (meaning that in preparing plates we arrange materials with any, only as necessary, set of material properties $(E, \nu) \in [E_0, E^0] \times [\nu_0, \nu^0]$) do not occur. In fact, in order to create plates materials may be taken from a certain finite set. We designate in terms of E_α and ν_α Young's modulus and Poisson's ratio for the α-th material of this set, $\alpha = 1, \ldots, M$, where M is total number of materials which may be used for preparing plates. In the case described, functions E(y) and $\nu(y)$ in (19) take values from set $\{(E_\alpha, \nu_\alpha), \alpha = 1, \ldots, M\}$ (discrete values) and pertain to a class of functions $U_d = \{f \in L_\infty([-1, 1]): f(t) \in \{(E_\alpha, \nu_\alpha), \alpha = 1, \ldots, M\}\ \forall t \in [-1, 1]\}$.

We present a method of approximate solution of Eq. (19) in class U_d. We carry out partition of the range [−1, 1] [for which integration is carried out in (19)] with a step of $\delta > 0$. We obtain partition $\Delta_n = [-1 + \delta n, -1 + \delta(n + 1))$, $n = 1, \ldots, [2/\delta]$. We introduce the function $y_\delta(y) = y_n = \delta n$ into $[-1 + \delta n, -1 + \delta(n + 1))$. We consider, together with (19), the problem

$$\left(\frac{h}{2}\right)^{\mu+\nu-1}\int_{-1}^{1}E(y)\,\Phi_{ijkl}(\nu(y))\,(y_\delta(y))^{\mu+\nu-2}\,dy=S_{ijkl}^{\mu\nu}. \tag{30}$$

It is easy to prove that for functions E(y), $\nu(y)$, and $\{\Phi_{ijk\ell}(\nu)\}$ the difference in the right-hand parts of (19) and (30) does not exceed the value

$$2\delta\max_{\alpha=1,\ldots,M}\frac{E_\alpha}{1-\nu_\alpha^2}, \tag{31}$$

i.e., the solution of problem (30) gives an approximate solution of (19) [the error in solution is estimated by the value of (31)].

We designate in terms of $\mu_\alpha{}^n$ a measure of a subset for the range Δ_n in which functions E(y) and $\nu(y)$ take the α-th value E_α, ν_α. In the notations adopted problem (30) leads to the form

$$\sum_{n=1}^{N}\sum_{\alpha=1}^{M}\mathbf{V}_n^\alpha\mu_\alpha^n=\mathbf{V},\quad \mathbf{V}_n^\alpha=\begin{pmatrix}\mathbf{Z}_n^\alpha\\ \mathbf{e}_n\end{pmatrix},\quad \mathbf{V}=\begin{pmatrix}\mathbf{Z}\\ \delta\mathbf{1}\end{pmatrix}, \tag{32}$$

where vector-columns $\mathbf{Z}_n^\alpha$, $\mathbf{Z}$, $\mathbf{e}_n$, $\mathbf{1}$ are

$$\mathbf{Z}_n^\alpha=\left((\delta n)^{\mu+\nu-1}E_\alpha\Phi_{ijkl}(\nu_\alpha)\right),\quad \mathbf{Z}=\left(S_{ijkl}^{\mu\nu}\left(\frac{2}{h}\right)^{\mu+\nu-1}\right)\in R^9,$$

$$\mathbf{e}_i=\begin{Bmatrix}0\\ \vdots\\ 1\\ \vdots\\ 0\end{Bmatrix}\ \text{in the i-th place},\ \mathbf{1}=\begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix}\in R^{[2/\delta]}.$$

In this way the relationship emerging from determining $\{\mu_n{}^\alpha\}$

$$\sum_{n=1}^{N} \sum_{\alpha=1}^{M} \mu_n^\alpha = 1, \quad \mu_n^\alpha \geqslant 0, \tag{33}$$

and also taking account of $\sum_{\alpha=1}^{M} \mu_n^\alpha = \delta$ the second group of equalities in (32) are fulfilled.

Problem (32), (33) is a problem of finding coefficients $\{\mu_n^\alpha\}$ for a convex combination of points $\{\mathbf{V}_n^\alpha\}$ supplying a given point $\mathbf{V}$. A study of this problem was carried out in [3], and an algorithm for numerical solution is given in [7]. According to [3], the set of solutions for problem (32), (33) (if it is not void) may be presented in the form

$$\mu_n^\alpha = \sum_{\gamma=1}^{s} P_{n\gamma}^\alpha \lambda_\gamma, \quad \sum_{\gamma=1}^{s} \lambda_\gamma = 1, \quad \lambda_\gamma \geqslant 0. \tag{34}$$

The method for finding values $\{P_{n\gamma}^\alpha\}$ (the so-called simplicial solutions) is given in [7, 8].

Construction of Particular Solutions of the Design Problem. In a number of cases the requirement does not arise of obtaining all of the possible solutions for the design problem, which occurs in using a small number of materials for creating a plate. Here a method of constructing particular solutions of problem (18) given below appears to be useful.

We shall build an approximate solution of problem (18). We perform discretization for the integration range $[-1, 1]$ with step $\delta > 0$ and for range $[H_1, H_2]$ (possible plate thicknesses) with step $\delta_1 > 0$. We introduce a piecewise-constant function $y_\delta(y) = y_n = -1 + n\delta$ with $y \in [-1 + n\delta, -1 + (n + 1)\delta)$, $i = 1, \ldots, [2/\delta]$ and a class of piecewise-constant functions

$$V = \left\{ \begin{array}{l} f \in L_\infty([-1, 1]) : f(y) = \text{const} \in \{E_\alpha\}_{\alpha=1}^{M} \\ \text{for } y \in [-1 + n\delta, -1 + (n+1)\delta), \ n = 1, \ldots, [2/\delta] \end{array} \right\}.$$

We consider the equation

$$H_j^{\mu\nu} \int_{-1}^{1} [y_\delta(y)]^{\mu+\nu-2} E(y) \Phi_{ijkl}(\nu(y))\, dy = S_{ijkl}^{\mu\nu}, \tag{35}$$

where $\mu, \nu = 1, 2$; $H_j^{\mu\nu} = ((H_1 + j\delta_1)/2)^{\mu+\nu-1}$; $E(y) \in V$.

As it is easy to prove, the solution of problem (35) satisfies equality (18) with an error not exceeding

$$\max\left(H_2, \frac{H_2^2}{8}, \frac{H_2^3}{12}\right) \max_{\alpha=1,\ldots,M} \left(E_\alpha \frac{\nu_\alpha}{1 - \nu_\alpha^2}\right)\delta. \tag{36}$$

Equation (35) in the given case is reduced to the following:

$$\delta \sum_{n=1}^{[2/\delta]} \mathbf{X}_{nj\alpha} = \mathbf{X}. \tag{37}$$

Here

$$\mathbf{X}_{nj\alpha} = \left(\left\{ \frac{H_j^{\mu\nu} E_\alpha y_n^{\mu+\nu-2}}{1 - \nu_\alpha^2} \right\}, \left\{ \frac{H_j^{\mu\nu} E_\alpha \nu_\alpha y_n^{\mu+\nu-2}}{1 - \nu_\alpha^2} \right\}, \left\{ \frac{H_j^{\mu\nu} E_\alpha y_n^{\mu+\nu-2}}{1 + \nu_\alpha} \right\} \right), \tag{38}$$

$$\mathbf{X} = (\{S_{1111}^{\mu\nu}\}, \{S_{1122}^{\mu\nu}\}, \{S_{1212}^{\mu\nu}\}), \quad \mu\nu = 11, 12, 22,$$

and indices $\alpha \in \{1, \ldots, M\}$ for terms in sum (37) may be assumed to be any values independently, and index $j \in \{1, \ldots, [(H_2 - H_1)/\delta_1]\}$ is the same for all terms. As can be seen from (37) and (38), presence in sum (37) of the term $\mathbf{X}_{nj\alpha}$ means that the n-th layer (layer $[-1 + n\delta, -1 + (n + 1)\delta]$) of the plate is occupied by the α-th material, and the plate thickness is $H_1 + j\delta_1$.

Problem (37) is incorrect [5]. With the aim of obtaining an approximate solution of it we shall perform minimization for the modulus of vector $S = X - \delta \sum_{n=1}^{[2/\delta]} X_{nj\alpha}$ [5] on the basis of the following algorithm:

1. In the first step a set of vectors $\{X^1_{nj\alpha}\} \subset \{X_{nj\alpha}\}$, in sum (37) is prescribed arbitrarily (e.g., $X^1_{nj\alpha} = X_{n11}$).

2. Let there be in the k-th step in sum (37) vectors $\{X^k_{nj\alpha}\}$. We carry out variation of the system $\{X^k_{nj\alpha}\}$: we substitute index j by j_* (which corresponds to a change in plate thickness from $H_1 + j\delta_1$ to $H_1 + j_*\delta_1$) and vector $X^k_{mj_*\alpha}$ by $X_{mj_*\alpha_*}$ (which relates to substitution of the m-th layer of the α-th material by the α_*-th). It is easy to calculate that the corresponding variation of vector S^k will equal

$$\delta S^k = \delta\left(\sum_{n=1}^{[2/\delta]} X_{nj_*\alpha} - X_{mj_*\alpha_*}\right) + \delta X_{mj_*\alpha_*} - \delta \sum_{n=1}^{[2/\delta]} X_{nj\alpha}. \tag{39}$$

If $|S^k + \delta S^k| < |S^k|$, the system of vectors is substituted by $\{X^k_{1j_*\alpha}, \ldots, X^k_{mj_*\alpha_*}, \ldots\} = \{X^{k+1}_{nj\alpha}\}$. With fulfillment of condition $|S^k + \delta S^k| < |S^k|$ the variation should be continued.

Note 3. Condition $|S^k + \delta S^k| < |S^k|$ may appear to be unfilfilled for any variations. This agrees with the fact that problem (37) is incorrect and it may not have a solution.

3. We carry out the actions described in 2) until, or not, we get into the situation in Note 3 (no solution) or discrepancy $S^k|$ becomes less than a prescribed value (a solution of the problem is obtained with a prescribed accuracy and it gives a set of vectors $\{X^k_{nj\alpha}\}$).

Example 1. Let it be required to obtain a plate design with stiffness characteristics: stiffness in tension $S^{11}_{1111} = 1\cdot10^9$, the obliquely symmetrical part of stiffness $S^{12}_{1111} = S^{21}_{1111} = 0$, stiffness in bending $S^{22}_{1111} = 0.06\cdot10^5$. It is noted that for a uniform plate with stiffness in tension $S^{11}_{1111} = 1\cdot10^9$ the rest of the characteristics listed above equal $S^{12}_{1111} = S^{21}_{1111} = 0$, $S^{22}_{1111} = 0.0937\cdot10^5$ ($\nu = 1/3$).

In order to solve the problem use was made of the algorithm described above for constructing a partial solution (realized in the form of a computer program in FORTRAN language). The plate design will be described by an equation consisting of fragments in the form nI (I is material index, n is the number of neighboring layers occupied by the I-th material). The following were selected as materials which may be used for preparing plates: Steel ($E = 2\cdot10^{11}$ Pa, $\nu = 0.3$, index St), aluminum ($E = 0.7\cdot10^{11}$ Pa, $\nu = 0.3$, index A), and capron ($E = 0.06\cdot10^{11}$ Pa, $\nu = 0.4$, index C). The condition for plate thickness $1 \le h \le 0.81\cdot10^{-2}$ m.

The computer produced a design (5C 18 St 1C 4 St 22C 1 St 1C 2St 1C 19 St 6C) (the plate through the thickness was broken down into 80 intervals with a step $\delta = h/80$). Plate thickness $h = 0.81\cdot10^{-2}$ m. Stiffness characteristics for the design were as follows:

$$S^{11}_{1111} = 1.0052\cdot10^9,\quad S^{12}_{1111} = 0.0003\cdot10^7,\quad S^{22}_{1111} = 0.0602\cdot10^5$$

(in the calculation the relative error of the solution was 0.02).

Note 4. As can be seen, the design obtained develops a property of rapid oscillation of characteristics [in it there is a fragment (1 St 1C 2 St 1C)]. Presence of this property is connected with the incorrectness of the problem and it may be analyzed in detail on the basis of Eq. (34). We shall not carry out a mathematical study of this question here. It is only noted that the effect noticed means that a solution for the type of composite material (containing a large number of thin alternating layers) arises in a natural way from solving a problem of the type in question. Existence of solutions corresponding to composites was noted in [9] in optimum design problems.

Example 2. Let it be required to obtain with the same materials and with the same limitations on thickness a design for a plate with stiffness characteristics

$$S^{11}_{1111} = 1\cdot10^9,\quad S^{12}_{1111} = S^{21}_{1111} = 0,\quad S^{22}_{1111} = 0.12\cdot10^5.$$

The computer produced a design (20 St 41C 19 St) (the breakdown step was the same as in Example 1). Plate thickness h = 0.905 m.

The stiffness characteristics for the design obtained were as follows:

$$S^{11}_{1111} = 1.0028\cdot 10^9,\quad S^{12}_{1111} = 0.0001\cdot 10^7,\quad S^{22}_{1111} = 0.118\cdot 10^5$$

(relative error of the solution 0.02).

LITERATURE CITED

1. D. Caillerie, "Thin elastic and periodic plates," Math. Meth. Appl. Sci., No. 6 (1984).
2. A. G. Kolpakov and S. I. Rakin, "Problem of synthesizing a composite material of unidimensional construction with prescribed characteristics," Zh. Prikl. Mekh. Tekh. Fiz., No. 6 (1986).
3. A. G. Kolpakov and S. I. Rakin, "Problem of synthesizing a composite of unidimensional structure in a prescribed class of materials," in: Solid Mechanics, No. 78 [in Russian], IG Sib. Otd. Akad. Nauk SSSR, Novosibirsk (1986).
4. A. G. Kolpakov, "Effective stiffness of composite plates," Prikl. Mat. Mekh., 46, No. 2 (1982).
5. A. N. Tikhonov and V. Ya. Arsenin, Methods of Solving Incorrect Problems [in Russian], Nauka, Moscow (1986).
6. V. M. Alekseev, V. M. Tikhomirov, and S. V. Fomin, Optimum Control [in Russian], Nauka, Moscow (1979).
7. A. G. Kolpakov, "Average characteristics of laminated composites (numerical algorithm)," in: Numerical Methods for Solving Elasticity and Plasticity Theory Problems: Proc. 9th All-Union Conf, Inst. Teor. Prikl. Mekh., Sib. Otd., Akad. Nauk SSSR, Novosibirsk (1986).
8. B. D. Annin and A. G. Kolpakov, "Planning of laminated composites with prescribed stress-strength characteristics," Mekh. Komposit. Mater., No. 1 (1987).
9. K. A. Lur'e and A. V. Cherkaev, "Regularization of optimum design problems for inhomogeneous elastic bodies by means of composite materials," in: 5th All-Union Conf. on Theoretical and Applied Mechanics [in Russian], Nauka, Alma-Ata (1981).

HYPERSONIC FLOW OVER BLUNT EDGES AT LOW REYNOLDS NUMBERS

A. V. Botin, V. N. Gusev, and V. P. Provotorov

UDC 533.6.011.8:533.694.71/72

In a planned descent from orbit a space vehicle is subjected to intense heating due to air flow. At the same time, even for relatively low flight heights and low blunting radii of individual vehicle elements, and thus low local Reynolds number values, the shock wave in those regions can no longer be considered as a discontinuity upon which the Rankin–Hugoniot relationships are satisfied, and the effect of viscosity is no longer limited to a thin boundary layer. At hypersonic velocities, because of the high flow energy, such physicochemical processes as heterogeneous chemical reactions, dissociation, and excitation of oscillatory, rotational, and translational degrees of molecular freedom may become significant in the disturbed region. The initial source of information on this transitional region was experiment. Subsequently, numerical methods were used successfully for solution of the Boltzmann equation for a homogeneous gas, the most widespread being the direct statistical modeling or Monte Carlo method. However, upon consideration of physicochemical processes in air such studies have as a rule been performed only with the Navier–Stokes equations or models thereof, with slippage boundary conditions and a temperature discontinuity. There is no strict justification for the applicability of such equations, although many comparisons with experimental results and numerical calculations of the Boltzmann kinetic equation for a homogeneous gas show that the Navier–Stokes equations can be used successfully for study of hypersonic flows

Moscow. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 4, pp. 161-168, July-August, 1989. Original article submitted June 26, 1986.

